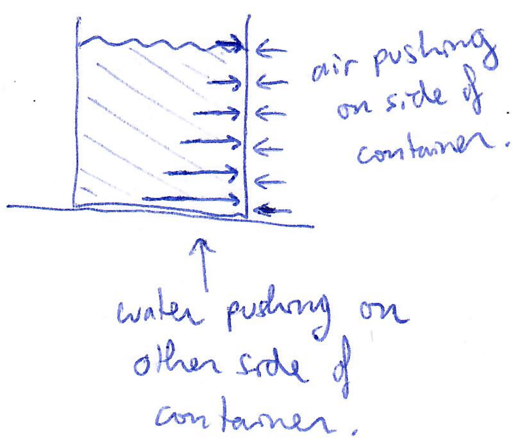
ME 4555 - Lecture 8 - Hydraulic systems

Hydraulic systems involve fluid flow, through pipes, reservoirs, pumps, valves, etc. as a means to store/transfer energy.

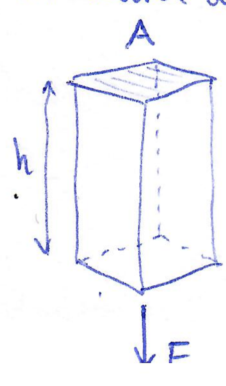
Some key properties of a small blob of fluid:

★ density (ρ): the ratio $\left(\frac{\text{mass}}{\text{volume}}\right)$. For fluids like water, we typically assume they are incompressible (ρ constant). Also a good assumption for air/gases moving at relatively low speed.

★ pressure (P): the force the fluid exerts over a given (per) area. This force is exerted in all directions. This is due to the adjacent fluid. On Earth, air pressure is due to the mass of air above us. Same for water pressure.



additional mass of water causes extra weight (and force) at lower depths.

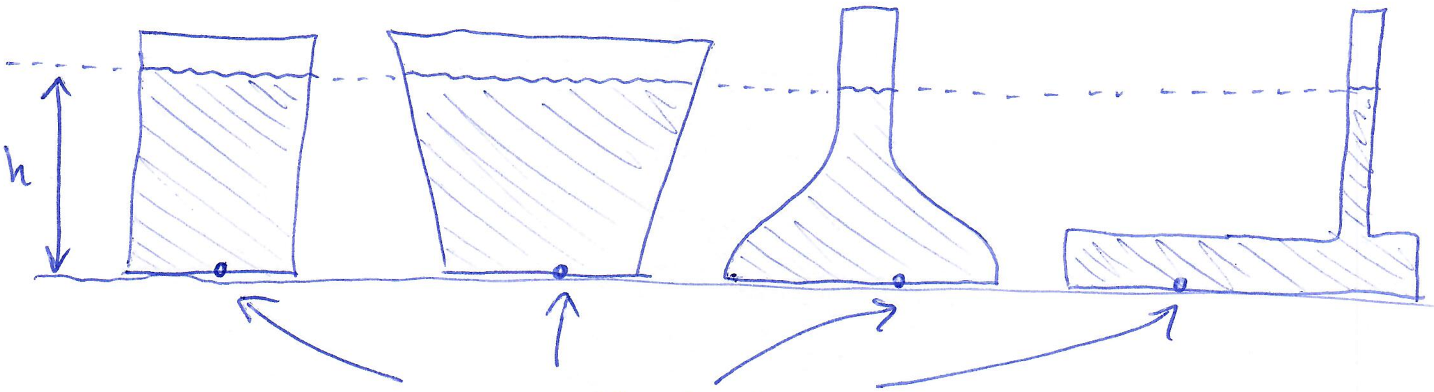


$$P = \frac{F}{A} = \frac{mg}{A} = \frac{\rho \cdot A \cdot h \cdot g}{A}$$

$$\Rightarrow \boxed{P = \rho g h}$$

due to gravity acting on a column of water.

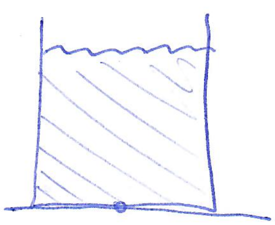
Pressure does <u>not</u> depend on the shape of the container. Only the depth!



pressure at all of these points is $\rho g h$.

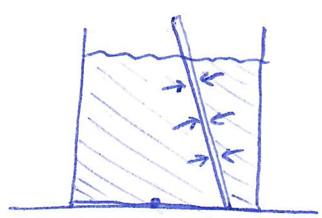
Why? Thought experiment.

1)



$P = \rho g h$

2) Add wall (soft/flexible)

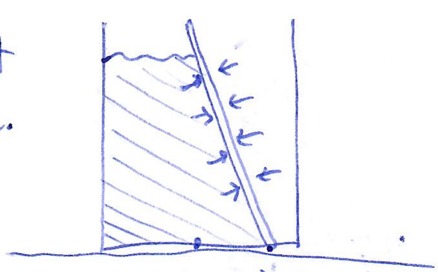


$P = \rho g h$

force is equal on both sides.

3) Remove top part to right of wall.

Replace pressure force with a reaction force.



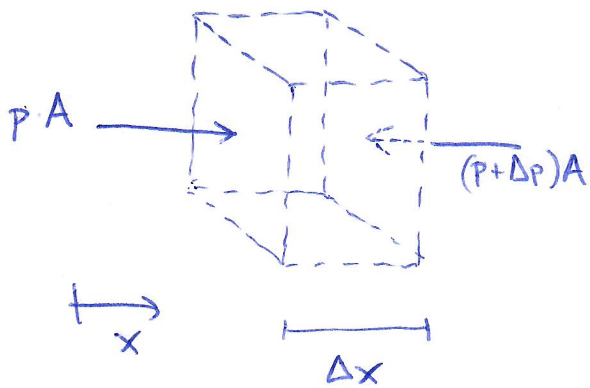
$P = \rho g h$

the liquid doesn't "know" that we did this!

how can anything change as a result of the new geometry?

Applying Newton's 2^{nd} law to a fluid:

Consider a blob of fluid (rectangular prism for simplicity) and consider 1-D motion of this blob in the horizontal direction.



say position is Δx wide, lateral area A, and pressure changes from p (left side) to $p + \Delta p$ (right side). Mass of blob is m, density (constant) is ρ.

$$m\ddot{x} = (p \cdot A) - (p + \Delta p)A.$$

$$\Rightarrow \boxed{m\ddot{x} + A\,\Delta p = 0}$$

$$\rho = \frac{m}{V} = \frac{m}{A\,\Delta x}$$

$$\Rightarrow \boxed{\rho\,\ddot{x}\,\Delta x + \Delta p = 0}$$

$$\ddot{x}\,\Delta x \approx \frac{\overset{\text{velocity}}{\Delta v}}{\underset{\text{time}}{\Delta t}} \cdot \Delta x \approx v\,\Delta v$$

$$\Rightarrow \boxed{\rho\, v\,\Delta v + \Delta p = 0}$$

let $\Delta v \to dv$ and $\Delta p \to dp$ <u>integrate</u>!

$$\Rightarrow \boxed{\frac{1}{2}\rho v^2 + p = \text{constant}}$$

this is called <u>Bernoulli's principle</u>.

Can also be interpreted as "kinetic energy + potential energy = constant".

NOTE

Equations of motion can usually be integrated to reveal some sort of energy balance. For example:

$$m\ddot{x} + c\dot{x} + kx = f.$$

multiply by dx.

$$m\ddot{x}\,dx + c\dot{x}\,dx + kx\,dx = f\,dx.$$

$$\begin{cases} m\ddot{x}\,dx = m\dfrac{dv}{dt}\cdot dx = mv\,dv \\ c\dot{x}\,dx = c\dfrac{dx}{dt}\cdot dx = c\dfrac{dx}{dt}\cdot\dfrac{dx}{dt}\cdot dt = cv^2\,dt. \end{cases}$$

$$\to \quad mv\,dv + cv^2\,dt + kx\,dx = f\,dx.$$

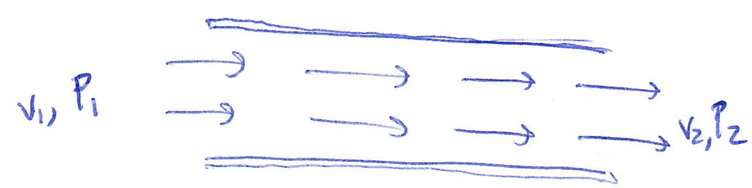
integrate!

$$\underbrace{\left[\frac{1}{2}mv^2\right]_{t_0}^{t_1}}_{\text{change in kinetic energy}} + \underbrace{c\int_{t_0}^{t_1} v^2\,dt}_{\substack{\text{energy loss due to friction} \\ \text{(always nonnegative)}}} + \underbrace{\left[\frac{1}{2}kx^2\right]_{t_0}^{t_1}}_{\text{change in (spring) potential energy}} = \underbrace{\int_{t_0}^{t_1} f\,dx.}_{\text{work done by external forces}}$$

In Lagrangian mechanics, equations of motion are derived using this approach in reverse (writing an energy balance and differentiating it, roughly).

Types of Flows.

1) inviscid (no viscosity / friction).



- if $P_1 = P_2$ (equal pressure) then $v_1 = v_2$ (equal velocity)
- if $P_1 > P_2$ then flow must accelerate by Bernoulli's principle:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

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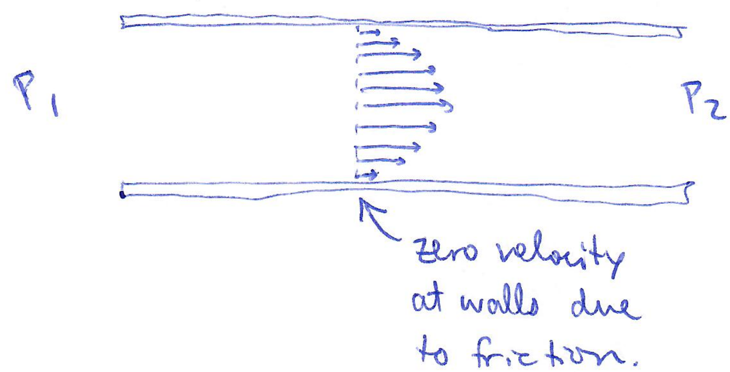
2) viscous effect (pipe is very long compared to diameter) flow is laminar.



- $P_1 > P_2$ required even to produce constant flow speed (Bernoulli does not hold)

$$\underbrace{P_1 - P_2}_{\text{pressure drop.}} = \overbrace{R}^{\text{"resistance"}} \cdot \underbrace{Q}_{\text{volumetric flow rate: } \dot{V}}$$

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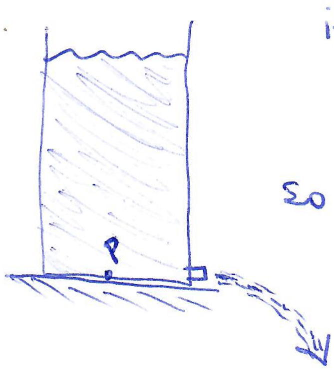
3) Nozzle / small opening: flow is turbulent.



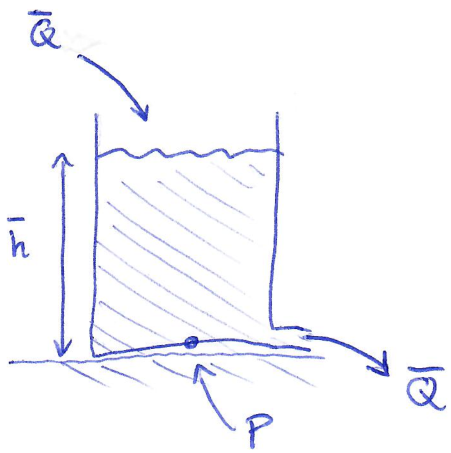
if ideal Bernoulli: $\underbrace{P}_{\rho g h} = \frac{1}{2}\rho v^2$ so $v = \sqrt{2gh}$

or $v = \sqrt{\frac{2}{\rho}} \cdot \sqrt{\Delta P}$

so $Q = v \underset{\text{area of opening}}{A} = A\sqrt{\frac{2}{\rho}\Delta P}$

In reality: $Q = \underset{\text{discharge coefficient.}}{C_d} \cdot A\sqrt{\frac{2}{\rho}\Delta P}$. i.e. $\boxed{Q \propto \sqrt{\Delta P}}$

Ex Single tank.



AT EQUILIBRIUM:

$\bar{Q}$ = inflow rate = outflow rate. (volume per second)

$\bar{h}$ = height of liquid level.

Using nozzle/small opening equation,

outflow = $\boxed{\bar{Q} = C \cdot \sqrt{P}.}$

Pressure due to depth: $\boxed{P = \rho g \bar{h}}$

If system conserves mass (it does!) then because density is constant, it also conserves volume.

$$Q_{in} - Q_{out} = \dot{V} = \dot{h} A. \Rightarrow \boxed{Q_{in} - Q_{out} = A\dot{h}}$$

Since we are at equilibrium, $Q_{in} = Q_{out}$, $\dot{h} = 0$.

So: $\bar{Q} = C\sqrt{\rho g \bar{h}}$

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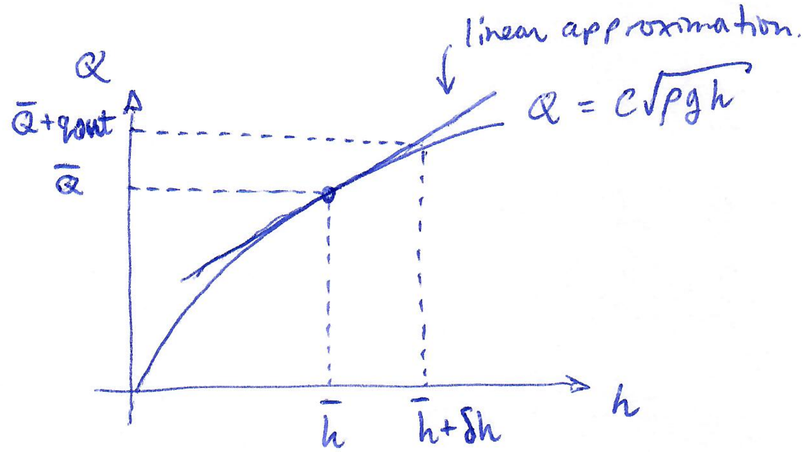
what if we perturb the inflow; i.e. $\begin{cases} Q_{in} = \bar{Q} + \underbrace{q_{in}}_{\text{small.}} \\ Q_{out} = \bar{Q} + q_{out}. \end{cases}$

if $q_{in} > 0$ then p increases,

which causes q_{out} to increase.



if δh is small compared to $\bar{h}$, the relationship is nearly linear (to good approximation).

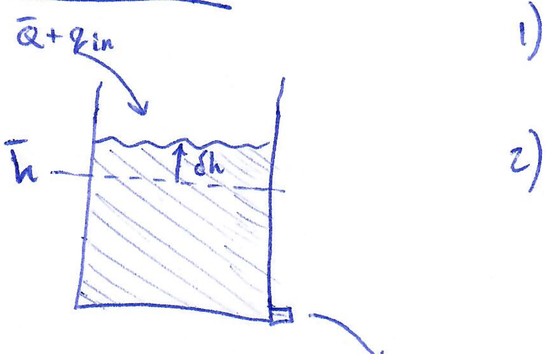
By Taylor's theorem, $\bar{Q} + q_{out} \approx \underbrace{Q\big|_{\bar{h}}}_{\bar{Q}} + \underbrace{\frac{dQ}{dh}\Big|_{\bar{h}}}_{\frac{c\sqrt{\rho g}}{2\sqrt{\bar{h}}}} \cdot \delta h$

So $q_{out} \approx \underbrace{\left(\frac{c}{2}\sqrt{\frac{\rho g}{\bar{h}}}\right)}_{\text{call this } \frac{1}{R}} \cdot \delta h.$

R is the "hydraulic resistance characteristic" of the pipe.

Therefore, $\boxed{q_{out} = \frac{\delta h}{R}}$

Summary

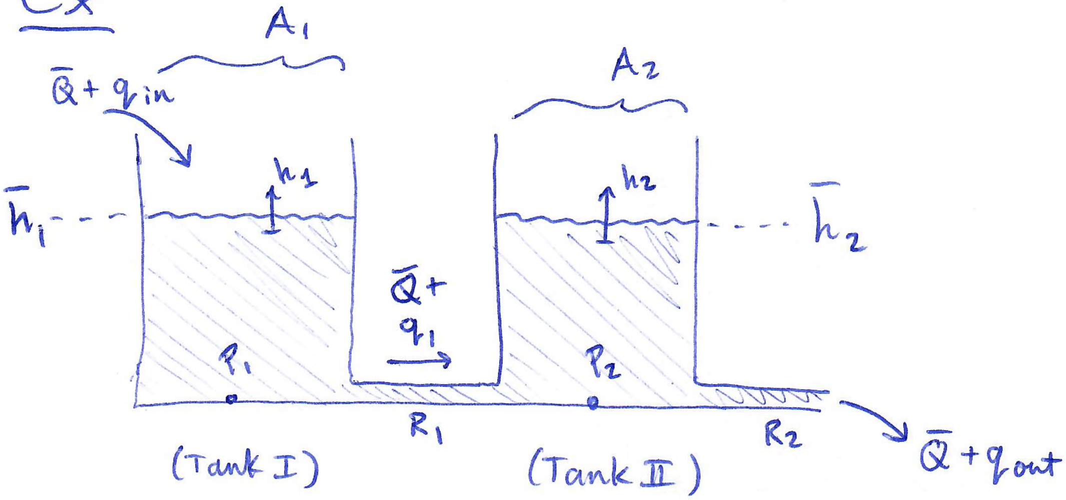


1) Conservation of mass: $q_{in} - q_{out} = A\dot{\delta h}$

2) Pipe flow: $q_{out} = \frac{\delta h}{R}$.

$\Rightarrow \boxed{A\dot{\delta h} + \frac{1}{R}\delta h = q_{in}}$

Ex



Tank I

$$q_{in} - q_1 = A_1 \dot{h}_1$$

$$q_1 = \frac{h_1 - h_2}{R_1}$$

Tank II

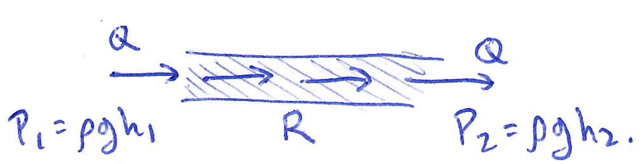
$$q_1 - q_{out} = A_2 \dot{h}_2$$

$$q_{out} = \frac{h_2}{R_2}$$

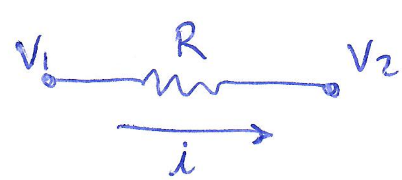
Combining equations and eliminating q_1, we get:

$$\begin{cases} A_1 \dot{h}_1 + \frac{1}{R_1} h_1 - \frac{1}{R_1} h_2 = q_{in} \\ A_2 \dot{h}_2 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) h_2 - \frac{1}{R_1} h_1 = 0 \end{cases}$$

Note: pipes with flowing liquids behave similarly to current flowing through a resistor.



$$\boxed{Q = \frac{h_1 - h_2}{R}}$$



$$\boxed{i = \frac{V_1 - V_2}{R}}$$